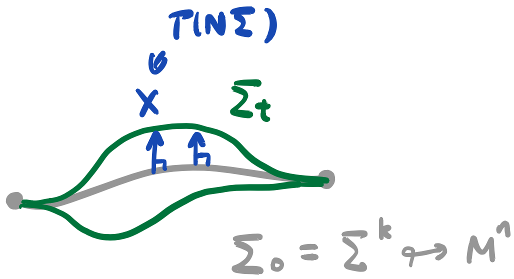
MATH 6021 Lecture 2 9/14/2020

Last time ......



Variation Formula

$$\delta\Sigma(X) = \int_\Sigma \mathrm{div}_\Sigma X \, dV = -\int_\Sigma \langle X, \vec{H} \rangle \, dV$$

$$\delta^2\Sigma(X,X) = -\int_\Sigma \langle X, L X \rangle \, dV \quad \text{—— } (*)$$

where $L : T(N\Sigma) \longrightarrow T(N\Sigma)$ Jacobi operator

$$L(X) := \Delta_\Sigma^N X + \sum_{i=1}^k \left(Rm_M(E_i, X) E_i \right)^N + \sum_{i,j=1}^k \langle A_{ij}, X \rangle A_{ij}$$

Note: $\{E_i\}$ O.N.B. for $T\Sigma$

Note: $A_{ij} = (\nabla_{E_i} E_j)^N$

Codimen 1 case $(k = n-1)$ (Assume 2-sided, $N\Sigma$ is trivialized by E_n)

$$\boxed{L\varphi := \Delta_\Sigma \varphi + Ric_M(E_n, E_n)\varphi + \|A\|^2 \varphi}$$

Def$^{\underline{n}}$: $\Sigma^k \subset M^n$ stable $\Longleftrightarrow$ $\lambda_1(L) \geq 0$ on cpt subsets $\Omega \subset \Sigma$.

Proof of (*):

Setup: $F : \Sigma^k \times (-\varepsilon, \varepsilon) \longrightarrow M^n$

$(x_1, \ldots, x_k)$: local coord. on Σ

$g(t) = g_{ij}(t) := \langle F_{x_i}, F_{x_j} \rangle$

$$v(t) = \frac{\sqrt{\det g(t)}}{\sqrt{\det g(0)}}$$

Assume:

(i) $X = F_t \big|_{t=0} \perp \Sigma$

(ii) $g_{ij}(0) = \delta_{ij}$ at $p \in \Sigma$

Note:

$$\dot{g}_{ij}(0) = \langle \nabla_{F_t} F_{x_i}, F_{x_j} \rangle + \langle F_{x_i}, \nabla_{F_t} F_{x_j} \rangle$$

$$= \langle \nabla_{F_{x_i}} X, F_{x_j} \rangle + \langle F_{x_i}, \nabla_{F_{x_j}} X \rangle$$

$$\overset{(i)}{=} -2 \langle A_{ij}, X \rangle$$

Recall: $|\Sigma_t| = \int_{\Sigma} \sqrt{\det g(t)}\, dx = \int_{\Sigma} \nu(t)\, dV_{\Sigma_0}$

Goal: Compute $\nu''(0)$ at $p \in \Sigma$.

(Note: $\underline{\nu'(0) \equiv 0}$ since Σ is minimal)

Recall: $\nu'(t) = \frac{1}{2} \frac{\sqrt{\det g(t)}}{\sqrt{\det g(0)}} \frac{d}{dt}\left(\log \det g(t)\right)$

$$= \sum_{i,j=1}^{k} \underline{g^{ij}(t)} \, \underline{\langle \nabla_{F_{x_i}} F_t , F_{x_j} \rangle (t)} \, \underline{\nu(t)}$$

Direct computation gives: At $t=0$, at $p \in \Sigma$ (Recall: ∇ connection on (M^n, g))

$$\nu''(0) = \underline{\sum_{i,j=1}^{k} \dot{g}^{ij}(0) \langle \nabla_{E_i} X , E_j \rangle} + \underline{\sum_{i=1}^{k} \langle \nabla_X \nabla_{E_i} X , E_i \rangle + \| \nabla_{E_i} X \|^2}$$

Term by term calculation:

- $\dot{g}^{ij}(0) = - \dot{g}_{ij}(0) = 2 \langle A_{ij} , X \rangle$
- $\langle \nabla_X \nabla_{E_i} X , E_i \rangle = \underbrace{\langle \nabla_{E_i} \nabla_X X , E_i \rangle}_{\sum_i (\ ) = \operatorname{div}_{\Sigma}(\nabla_X X)} - \langle Rm_M (X, E_i) X , E_i \rangle$
- $\| \nabla_{E_i} X \|^2 = \| \nabla^N_{E_i} X \|^2 + \underbrace{\| \nabla^T_{E_i} X \|^2}_{\| \sum_{j=1}^{k} \underbrace{\langle \nabla_{E_i} X , E_j \rangle}_{(ij) \ = -\langle A_{ij} , X \rangle} E_j \|^2}$

Putting it all together, we have proved (*). ∎

Digression: Calibrated Geometry [Harvey-Lawson '82]

Note: There is a useful tool to check a min. submfd minimizes area in some class, called "calibration argument".

Let $\mathcal{U} \subseteq (M^n, g)$ be an open subset, and $\Sigma^k \subseteq \mathcal{U}$ be an oriented submfd.

Def^n: Σ^k is calibrated by a k-form $\omega \in \Omega^k(\mathcal{U})$ if

① $d\omega = 0$, ie. ω is a closed form in $\mathcal{U}$.

② $\omega|_{T\Sigma}$ = area form on Σ

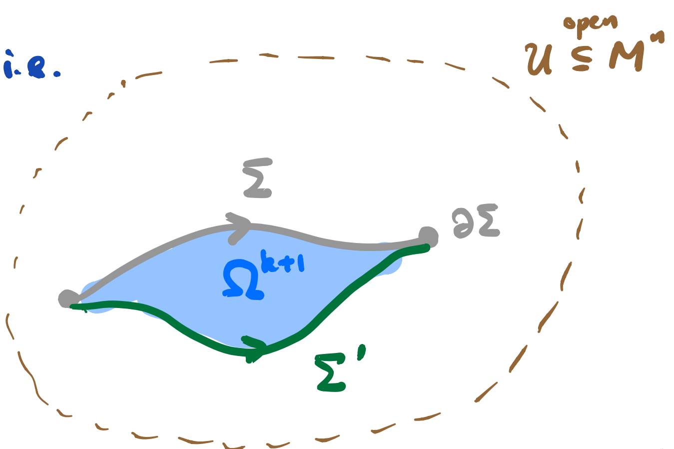
③ For any other oriented submfd $\Sigma' \subseteq \mathcal{U}$.

$$\omega|_{T\Sigma'} \leq \text{area form of } \Sigma'$$

Calibration Lemma:

$\Sigma^k \subseteq \mathcal{U}$ calibrated by $\omega \Rightarrow \Sigma$ is "homologically area-minimizing" inside $\mathcal{U}$.

i.e.



$$\partial\Omega = \Sigma' - \Sigma$$

If $\Sigma' \subseteq \mathcal{U}$ is any oriented k-submfd with $\partial\Sigma' = \partial\Sigma$ and $\partial\Omega = \Sigma' - \Sigma$ for some $(k+1)$-oriented submfd $\Omega \subseteq \mathcal{U}$.

then $|\Sigma'| \leq |\Sigma|$.

Proof: By Stokes' Thm.

$$0 \overset{①}{=} \int_\Omega d\omega \overset{\downarrow}{=} \int_{\partial\Omega} \omega = \overbrace{\int_{\Sigma'} \omega|_{T\Sigma'}}^{|\Sigma'| \ \overset{③}{\geq}} - \overbrace{\int_{\Sigma} \omega|_{T\Sigma}}^{|\Sigma| \ \overset{②}{=}}$$

$\blacksquare$

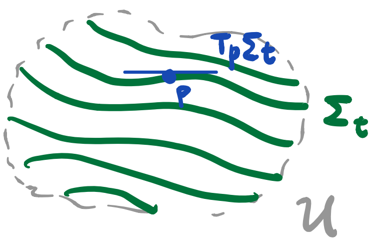
Applications

(I) **Thm:** All complex submfd of $\mathbb{C}^n$ are homologically area-minimizing.

Reason: Kähler form $\omega \rightsquigarrow \omega^m$ calibrates cpx submfd Σ^{2m}.

(II) **Prop:** If $\mathcal{U} \subseteq M^n$ is "foliated" by min. hypersurface, i.e. $\mathcal{U} = \bigcup_{t \in I} \Sigma_t$

then each leaf Σ_t^{n-1} is homologically area-min in $\mathcal{U}$.



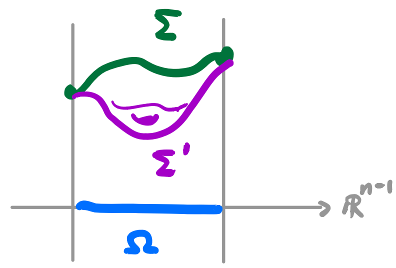
Reason: Define ω_p = area form of the leaf Σ_t passes through p

②, ③ trivial

① follows $d\omega = H_{\Sigma_t}\, d\mathrm{Vol}_M = 0$.

(III) **Cor 1:** $\Sigma^{n-1} \subseteq \mathbb{R}^n$ min. graph over $\Omega \subseteq \mathbb{R}^{n-1}$

$\Rightarrow$ Σ "homological" area-minimizing in $\Omega \times \mathbb{R}$



Reason: Translates of Σ gives a min. foliation of $\Omega \times \mathbb{R}$

In particular, if Σ is an entire min. graph (ie $\Omega = \mathbb{R}^{n-1}$)

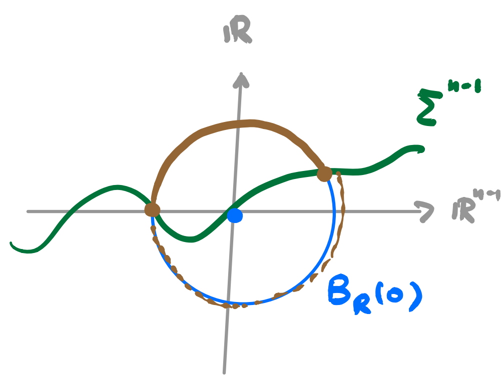
$\Rightarrow$ Σ is area-minimizing in $\mathbb{R}^n$ (even among non-graphical competitors)

This, implies the following Euclidean Volume Growth

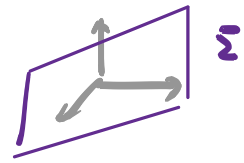
Cor 2: $|\Sigma \cap B_R(0)| \leq C R^{n-1}$

for some constant $C > 0$.

by comparing with region on $\partial B_R(0)$



Bernstein-type Results



Bernstein Theorem (Bernstein ~1915)

Any entire minimal graph in $\mathbb{R}^3$ is a plane.

i.e. If $u : \mathbb{R}^2 \to \mathbb{R}$ solves the (MSE), then

$$u(x,y) = ax + by + c \quad \text{for } a, b, c \in \mathbb{R} \text{ constants}$$

Remarks: (i) Different from entire harmonic functions $\Delta u = 0$

e.g. $u(x,y) = x^2 - y^2$ on $\mathbb{R}^2$

(ii) The same theorem holds up to $\mathbb{R}^n$, $n \leq 8$.

[Fleming '62, De Giorgi '65, Almgren '66, Simons '68]

(iii) Thm is false in dim. $n > 8$ [Simons '68, Bombieri-De Giorgi-Giusti '69]

(iv) This is closely related to regularity theory of min surfaces.

A "Geometric" Proof of Bernstein Thm $(n=3)$ - L. Simon

Let $\Sigma = \text{graph}(u) \subseteq \mathbb{R}^3$ be an entire min. graph.

① Σ minimizing $\Rightarrow$ stable $\Rightarrow$ "Stability inequality"

$$\boxed{\int_\Sigma |A|^2 \varphi^2 \leq \int_\Sigma |\nabla \varphi|^2, \quad \forall \varphi \in C_c^\infty(\Sigma)} \quad -(1)$$

Calibration argument $\Rightarrow$

② "Euclidean Volume Growth"

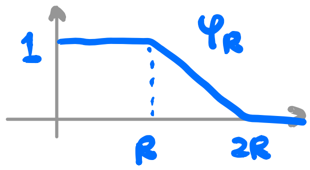
$$\boxed{|\Sigma \cap B_R(0)| \leq C R^2 \quad \forall R > 0} \quad -(2)$$

<u>Idea:</u> Choose a seq. of cutoff functions φ_i on Σ s.t.

- $\varphi_i \to 1$ uniformly on cpt subsets of Σ
- $\int_\Sigma |\nabla \varphi_i|^2 \longrightarrow 0$ as $i \to \infty$

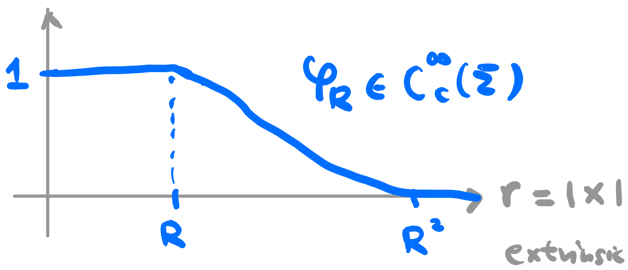
THEN: (1) gives $\int_\Sigma |A|^2 = 0 \Rightarrow |A| \equiv 0 \Rightarrow \Sigma$ flat.

<u>Naive Try</u>:



$|\nabla \varphi_R|^2 \sim \frac{1}{R^2} \Rightarrow \int_\Sigma |\nabla \varphi_R|^2 \leq C$

<u>Logarithmic cutoff trick</u>



extrinsic distance

[<u>Note</u>: $|\nabla r| \leq 1$]

$$\varphi_R(r) := \begin{cases} 1 & , \quad r \leq R \\ \frac{\log R^2/r}{\log R} & , \quad R \leq r \leq R^2 \\ 0 & , \quad r \geq R^2 \end{cases}$$

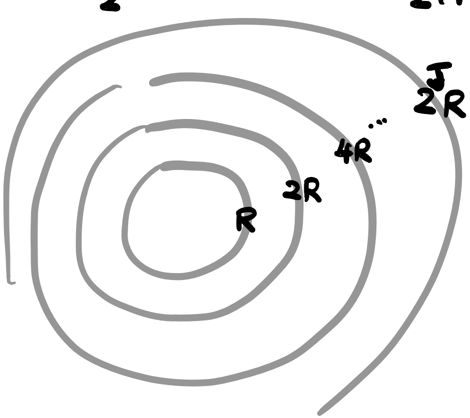
<u>Note</u>: $\frac{\partial}{\partial r}\varphi_R = -\frac{1}{r \log R}$

$|\nabla \varphi_R|^2 \sim \frac{1}{(R \log R)^2}$

<u>Note</u>: $\varphi_R \to 1$ uniformly on cpt subsets

Moreover,

$$\int_\Sigma |\nabla \varphi_R|^2 \lesssim \int_{\Sigma \cap (B_{R^2} \setminus B_R)} \frac{C}{r^2 (\log R)^2} = \frac{C}{(\log R)^2} \int_{\Sigma \cap (B_{R^2} \setminus B_R)} \frac{1}{r^2}$$



$2^J R = R^2$

$J \approx \log R$

$$= \frac{C}{(\log R)^2} \sum_{j=1}^{J} \int_{\Sigma \cap (B_{2^j R} \setminus B_{2^{j-1} R})} \frac{1}{r^2}$$

$$\leq \frac{C}{(\log R)^2} \sum_{j=1}^{J} \underbrace{\frac{1}{(2^{j-1} R)^2} |\Sigma \cap B_{2^j R}|}_{\lesssim 4C} \leq \frac{C'}{\log R}$$

$\square$

Q: Can one relax the hypothesis of being "entire min. graph"?

Stable Bernstein Conjecture ($n \leq 8$) still open even for $n=4$.

$\Sigma^{n-1} \subseteq \mathbb{R}^n$ stable, complete (immersed) min. hypersurface $\Longrightarrow$ Σ = hyperplane

Remarks:

(1) This conjecture is answer affirmatively under certain additional assumptions, e.g. Euclidean volume growth [$n=3,4,5$ Schoen-Simon-Yau '75], embeddedness [L. Simon '76]

(2) This is settled in full generality in dim $n=3$, independently by Fisher-Colbrie & Schoen '80 and Do Carmo-Peng '79.